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Replica analysis of the p-spin interaction Ising spin-glass model

Viviane M de Oliveira and J F Fontanari

Instituto de Física de São Carlos, Universidade de São Paulo, Caixa Postal 369, 13560-970 São Carlos SP, Brazil

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Abstract. The thermodynamics of the infinite-range Ising spin glass with p-spin interactions in the presence of an external magnetic field h is investigated analytically using the replica method. We give emphasis to the analysis of the transition between the replica symmetric and the one-step replica symmetry breaking regimes. In particular, we derive analytical conditions for the onset of the continuous transition, as well as for the location of the tricritical point at which the transition between those two regimes becomes discontinuous.

1. Introduction

Although the thermodynamics of the Ising spin glass with infinite-range interactions, the so-called Sherrington–Kirkpatrick (SK) model [1], has been thoroughly investigated in the last two decades [2, 3], comparatively little attention has been given to the analysis of a natural generalization of the SK model, namely, the p-spin interaction Ising spin glass. This model is described by the Hamiltonian [4, 5]

$$\mathcal{H}_p(\boldsymbol{S}) = -\sum_{1\leqslant i_1<i_2\ldots<i_p\leqslant N} J_{i_1 i_2\ldots i_p} S_{i_1} S_{i_2} \ldots S_{i_p} - h\sum_i S_i \tag{1}$$

where $S_i = \pm 1$, $i = 1, \ldots, N$ are Ising spins and h is the external magnetic field. Here the coupling strengths are statistically independent random variables with a Gaussian distribution

$$\mathcal{P}(J_{i_1 i_2\ldots i_p}) = \sqrt{\frac{N^{p-1}}{\pi p!}} \exp\left[-\frac{(J_{i_1 i_2\ldots i_p})^2 N^{p-1}}{p!}\right]. \tag{2}$$

Besides the acknowledged importance of the p-spin interaction Ising spin glass in the framework of the traditional statistical mechanics of disordered systems (it yields the celebrated random energy model in the limit $p \to \infty$ [4] and the SK model for $p = 2$), it also plays a significant role in the study of adaptive walks in rugged fitness landscapes within the research programme championed by Kauffman [6–8].

The thermodynamics of the SK model ($p = 2$) as well as that of the random energy model ($p \to \infty$) are now well understood. In the Parisi replica framework the local order parameter is a $n \times n$ matrix q_{ab}, which in the limit $n \to 0$ is characterized by a function $q(x)$ where $0 \leqslant x \leqslant 1$. The inverse function $x(q)$ has a clear physical interpretation: it is the probability that two equilibrium states have an overlap smaller than q [2, 3]. In particular, for $p = 2$ and $h = 0$ the order parameter function $q(x)$ tends to zero continuously as the temperature approaches the critical value $T_c^{(2)} = 1$ at which the transition between the spin glass and the high-temperature (disordered) phases takes place [2,3]. For $p \to \infty$ and $h = 0$, the system has a critical temperature $T_c^{(\infty)} = 1/(2\sqrt{\ln 2})$ at which it freezes completely into the ground state: $q(x)$ is a step function with values zero and one, and with a break point at $x = T/T_c^{(\infty)}$ [4, 5]. These results are not affected qualitatively by the presence of a non-zero magnetic field. More pointedly, for the SK model the critical temperature decreases monotonically with increasing h while the transition remains continuous, in the sense that $q(x)$ is continuous at the transition line [9, 10]. In contrast, for the random energy model the critical temperature increases with increasing h while the discontinuity in the step function $q(x)$ decreases with increasing h and vanishes in the limit $h \to \infty$ [4, 5].

The situation for finite $p > 2$ is considerably more complicated and so the thermodynamics of the p-spin model has been investigated for $h = 0$ only [11,12]. In this case there is a transition from the disordered phase to a partially frozen phase characterized by a step function $q(x)$ with values zero and $q_1 < 1$. As the temperature is lowered further, a second transition occurs, leading to a phase described by a continuous order parameter function [11,12]. However, there is evidence that the presence of a non-zero magnetic field decreases the size of the discontinuity of the order parameter $q(x)$ leading, eventually, to a continuous phase transition. In fact, a recent analysis of the typical overlap $\bar{q}$ between pairs of metastable states with energy density ϵ indicates that $\bar{q}$ is a discontinuous function of ϵ for $p > 2$, and that the size of the jump in $\bar{q}$ increases with p and decreases with h, vanishing at finite values of the magnetic field [13]. Moreover, a similar effect has already been observed in the thermodynamic analysis of the spherical p-spin interaction spin-glass model [14]. It is interesting to note that the spin-glass phase of this continuous spin model is described exactly by a step order parameter function, i.e., the one-step replica symmetry breaking (1RSB) is the most general solution within the Parisi scheme of replica symmetry breaking [14].

In this paper we use the replica method to study the thermodynamics of the Ising p-spin interaction spin-glass model in the presence of the magnetic field h. We focus on the effects of h on the transition between the replica symmetric (RS) and the 1RSB regimes. In particular, we show that for $p > 2$ the discontinuous transition reported in previous analyses [11, 12] turns into a continuous one for h larger than a certain value h_T. Moreover, we derive analytical conditions to determine the location of the continuous transition line, as well as that of the tricritical point at which the transition becomes discontinuous.

The remainder of this paper is organized as follows. In section 2 we discuss the replica formulation and present the formal equation for the average free-energy density, which is then rewritten using the RS and the 1RSB ansatze. These results are discussed very briefly since their derivations are given in detail in Gardner's paper [11]. We also present the solution of the 1RSB saddle-point equations in the limit of large p, thus extending the series expansions results for non-zero h. In section 3 we derive analytical conditions for locating the continuous transition and the tricritical point between the RS and 1RSB regimes, and present the phase diagrams in the plane (T, h). Finally, some concluding remarks are presented in section 4.

2. The replica formulation

We are interested in the evaluation of the average free-energy density f defined by

$$-\beta f = \lim_{N\to\infty} \frac{1}{N} \langle\langle \ln Z \rangle\rangle \tag{3}$$

where

$$Z = \mathrm{Tr}_{\boldsymbol{S}} \exp[-\beta \mathcal{H}_p(\boldsymbol{S})] \tag{4}$$

is the partition function and β is the inverse temperature. Here $\langle\langle\ldots\rangle\rangle$ stands for the average over the coupling strengths, and $\mathrm{Tr}_{\boldsymbol{S}}$ denotes the summation over the 2^N states of the system. As usual, the evaluation of the quenched average in equation (3) can be effectuated through the replica method: using the identity

$$\langle\langle \ln Z\rangle\rangle = \lim_{n\to 0}\frac{1}{n}\ln\langle\langle Z^n\rangle\rangle \tag{5}$$

we first calculate $\langle\langle Z^n\rangle\rangle$ for *integer* n, i.e. $Z^n = \prod_{a=1}^{n} Z_a$, and then analytically continue to $n = 0$ [2,3]. The final result is simply [11]

$$-\beta f = \lim_{n\to 0}\mathrm{extr}\left[\frac{1}{n}G(q_{ab},\lambda_{ab})\right] + \frac{1}{4}\beta^2 \tag{6}$$

where

$$G(q_{ab},\lambda_{ab}) = \tfrac{1}{2}\beta^2\sum_{a<b} q_{ab}^p - \beta^2\sum_{a<b}\lambda_{ab}q_{ab} + \ln \mathrm{Tr}_{\{S^a\}}\exp\left(\beta^2\sum_{a<b}\lambda_{ab}S^aS^b + \beta h\sum_a S^a\right). \tag{7}$$

The extremum in equation (6) is taken over the physical order parameter

$$q_{ab} = \left\langle\left\langle \frac{1}{N}\sum_{i=1}^{N}\langle S_i^a\rangle_T\langle S_i^b\rangle_T\right\rangle\right\rangle \qquad a < b \tag{8}$$

which measures the overlap between two different equilibrium states $\boldsymbol{S}^a$ and $\boldsymbol{S}^b$, and over its corresponding Lagrangian multiplier λ_{ab}. Here, $\langle\ldots\rangle_T$ stands for a thermal average. To proceed further, next we consider two standard ansatze for the structure of the saddle-point parameters.

2.1. RS solution

In this case we assume that the saddle-point parameters are symmetric under permutations of the replica indices, i.e., $q_{ab} = q$ and $\lambda_{ab} = \lambda$. With this prescription the evaluation of equation (6) is straightforward, resulting in the RS free-energy density

$$-\beta f_{rs} = -\tfrac{1}{2}\beta^2\lambda(1-q) + \tfrac{1}{4}\beta^2(1-q^p) + \int_{-\infty}^{\infty}\mathrm{D}z\ \ln 2\cosh[\beta\,\Xi_s(z)] \tag{9}$$

where

$$\Xi_s = z\sqrt{\lambda} + h \tag{10}$$

and

$$\mathrm{D}z = \frac{\mathrm{d}z}{\sqrt{2\pi}}\mathrm{e}^{-z^2/2} \tag{11}$$

is the Gaussian measure. The saddle-point equations $\partial f_{rs}/\partial q = 0$ and $\partial f_{rs}/\partial\lambda = 0$ yield

$$\lambda = \frac{p}{2}q^{p-1} \tag{12}$$

and

$$q = \int_{-\infty}^{\infty}\mathrm{D}z\ \tanh^2[\beta\,\Xi_s(z)] \tag{13}$$

respectively. The RS solution is locally stable wherever the Almeida–Thouless condition [9], which in this case is given by [11]

$$1 - \beta^2(p-1)\frac{\lambda}{q}\int_{-\infty}^{\infty}\mathrm{D}z\ \mathrm{sech}^4[\beta\,\Xi_s(z)] > 0 \tag{14}$$

is satisfied. In fact, since equation (13) has either one or three positive solutions, this stability condition is very useful to single out the physical one. In particular, for $h = 0$ the only stable solution is $q = 0$.

2.2. Replica symmetry broken solution

Following Parisi's scheme [3], we carry out the first step of replica symmetry breaking by dividing the n replicas into n/m groups of m replicas and setting $q_{ab} = q_1$, $\lambda_{ab} = \lambda_1$ if a and b belong to the same group and $q_{ab} = q_0$, $\lambda_{ab} = \lambda_0$ otherwise. The physical meaning of the saddle-point parameters is the following

$$q_0 = \left\langle\!\!\left\langle \frac{1}{N} \sum_{i=1}^{N} \langle S_i^a \rangle_T \langle S_i^b \rangle_T \right\rangle\!\!\right\rangle \qquad a < b \tag{15}$$

$$q_1 = \left\langle\!\!\left\langle \frac{1}{N} \sum_{i=1}^{N} \langle S_i^a \rangle_T^2 \right\rangle\!\!\right\rangle \tag{16}$$

and $m = 1 - \sum_a P_a^2$. Hence q_0 is the overlap between a pair of different equilibrium states, q_1 is the overlap of an equilibrium state with itself ($q_1 \geqslant q_0$), and m is the probability of finding two copies of the system in two different states (P_a is just the Gibbs probability measure for the state $\boldsymbol{S}^a$). We note that in the limit $n \to 0$, the parameter m is constrained to the range $0 \leqslant m \leqslant 1$. Using this prescription, equation (6) becomes

$$-\beta f_{rsb} = -\frac{1}{4}\beta^2[2\lambda_1(1 - q_1 + mq_1) - 2mq_0\lambda_0 - 1 + (1-m)q_1^p + mq_0^p] \\ + \ln 2 + \frac{1}{m}\int_{-\infty}^{\infty} \mathrm{D}z_0 \ln \int_{-\infty}^{\infty} \mathrm{D}z_1 \cosh^m \beta\,\Xi \tag{17}$$

where

$$\Xi = z_1\sqrt{\lambda_1 - \lambda_0} + z_0\sqrt{\lambda_0} + h. \tag{18}$$

The saddle-point equations $\partial f_{rsb}/\partial q_k = 0$ yield

$$\lambda_k = \frac{p}{2} q_k^{p-1} \tag{19}$$

for $k = 0, 1$. The saddle-point parameters q_0, q_1 and m are given by the equations

$$q_0 = \int_{-\infty}^{\infty} \mathrm{D}z_0\, \langle \tanh \beta\,\Xi \rangle_z^2 \tag{20}$$

$$q_1 = \int_{-\infty}^{\infty} \mathrm{D}z_0\, \langle \tanh^2 \beta\,\Xi \rangle_z \tag{21}$$

and

$$\frac{1}{4}\beta^2(p-1)(q_1^p - q_0^p) = -\frac{1}{m^2} \int_{-\infty}^{\infty} \mathrm{D}z_0 \ln \int_{-\infty}^{\infty} \mathrm{D}z_1 \cosh^m \beta\,\Xi + \frac{1}{m}\int_{-\infty}^{\infty} \mathrm{D}z_0\, \langle \ln\cosh \beta\,\Xi \rangle_z \tag{22}$$

where we have introduced the notation

$$\langle \ldots \rangle_z = \frac{\int_{-\infty}^{\infty} \mathrm{D}z_1\, (\ldots) \cosh^m \beta\,\Xi}{\int_{-\infty}^{\infty} \mathrm{D}z_1 \cosh^m \beta\,\Xi}. \tag{23}$$

It is clear from these equations that the RS saddle-point $q_0 = q_1 = q$ is a solution for any value of m. In general, however, the 1RSB equations will admit a different solution. In particular, in the limit $p \to \infty$ the solution is $q_0 = \tanh^2(\beta m h)$, $q_1 = 1$ and $m = \beta_c/\beta$ where $\beta_c = 1/T_c^{(\infty)}$ is the solution of the equation [5]

$$\tfrac{1}{4}\beta_c^2 = \ln 2 \cosh \beta_c h - \beta_c h \tanh \beta_c h. \tag{24}$$

Below $T_c^{(\infty)}$, the entropy vanishes and m sticks to its maximum value, namely $m = 1$, signalling the existence of a frozen phase in accordance with the physical meaning of m mentioned before. It is instructive to consider the finite p corrections to the infinite-p solution by expanding the 1RSB equations around that solution. Thus, extending the results of Gardner [11] for non-zero h, we find

$$q_0 = \tanh^2(\beta m h)\left[1 + 2\xi_m \operatorname{sech}(\beta m h)\frac{\mathrm{e}^{-\beta^2 m^2 p/4}}{\sqrt{\frac{1}{2}p\beta^2}}\right] \tag{25}$$

$$q_1 = 1 - \frac{m\xi_m}{1-m}\operatorname{sech}(\beta m h)\frac{\mathrm{e}^{-\beta^2 m^2 p/4}}{\sqrt{\frac{1}{2}p\beta^2}} \tag{26}$$

and

$$\frac{1}{4}\beta^2 = \frac{1}{m^2}[\ln 2\cosh(\beta m h) - \beta m h \tanh(\beta m h)] + \Lambda_m \tag{27}$$

where

$$\Lambda_m = -\sqrt{\tfrac{1}{2}p\beta^2}\,\xi_m \operatorname{sech}(\beta m h)\mathrm{e}^{-\beta^2 m^2 p/4} \tag{28}$$

if $\beta^2 m^2 < 8|\ln\tanh(\beta m h)|$ and

$$\Lambda_m = \beta^2 p \tanh^{2p}(\beta m h)\frac{\beta m h}{\sinh(2\beta m h)} \tag{29}$$

otherwise. Here,

$$\begin{aligned}\xi_m &= -\frac{1}{\sqrt{2\pi}}\sum_{i=0}^{\infty}\binom{m}{i}\frac{1}{2i-m}\\ &= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\mathrm{d}z\,[2\cosh(mz) - 2^m\cosh^m(z)]\end{aligned} \tag{30}$$

where we have used the extended definition of the binomial coefficient to real m [15].

At this stage we can already realize the existence of two solutions of a quite different nature, signalling then the nontrivial role played by the magnetic field in the thermodynamics of the p-spin model.

A quite interesting property of the 1RSB solution, which can easily be verified numerically, is that $q_0 = 0$ for $h = 0$ and $p > 2$, thus indicating that the equilibrium states are completely uncorrelated. Moreover, this result has greatly facilitated both the numerical and analytical analyses of the model, since the integrals over z_0 in equations (20)–(22) can be carried out trivially in that case [11, 12]. However, as explicitly shown by equation (25) the non-zero magnetic field induces correlations between different equilibrium states so that q_0 is no longer zero in this case.

For $p = 3$, we present in figures 1–3 the temperature dependence of the RS and 1RSB saddle-point parameters for $h = 0$, 0.5 and 1, respectively. As mentioned before, for $h = 0$ we find $q = q_0 = 0$. The size of the jump in q_1 decreases with increasing h and disappears altogether for $h \geqslant h_T^{(3)} \approx 0.57$. Of particular interest is the temperature dependence of the saddle-point parameter m: at the discontinuous transition it reaches its maximal value, namely, $m = 1$, while at the continuous transition it assumes a certain value $m = m_c \leqslant 1$,

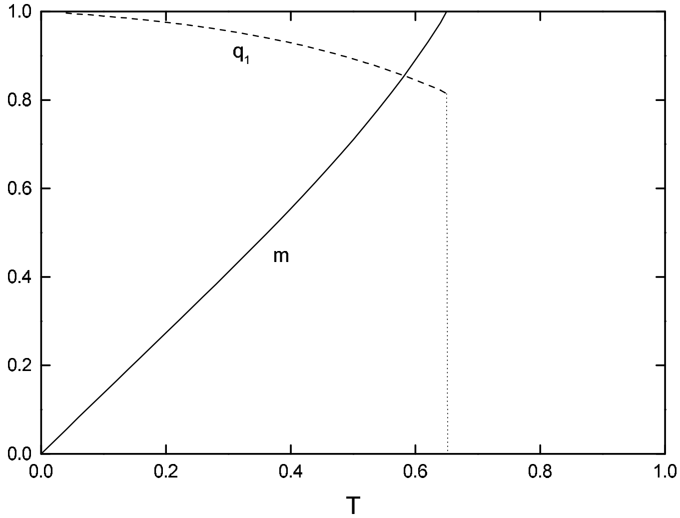

Figure 1. 1RSB saddle-point parameters m (full curve) and q_1 (short-broken curve) as a function of the temperature T for $p = 3$ and $h = 0$. In this case $q_0 = q = 0$. The discontinuous transition occurs at $T \approx 0.65$.

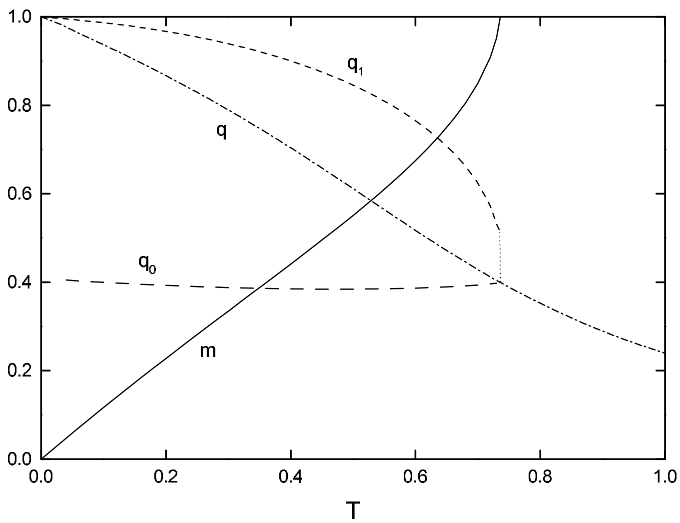

Figure 2. 1RSB saddle-point parameters m (full curve), q_0 (long-broken curve), q_1 (short-broken curve) as a function of the temperature T for $p = 3$ and $h = 0.5$. The chain curve is the RS saddle-point parameter q. The discontinuous transition occurs at $T \approx 0.74$.

which depends on T and p. As expected, the behaviour pattern depicted in figure 3 is very similar to that found in the analysis of the magnetic properties of the SK model [10], as the transition is continuous in that model. We note that since m plays no role in the RS solution, the curve for m must end at the transition lines.

The location of the transition lines as well as the characterization of the critical values of the saddle-point parameters are discussed in detail in the next section.

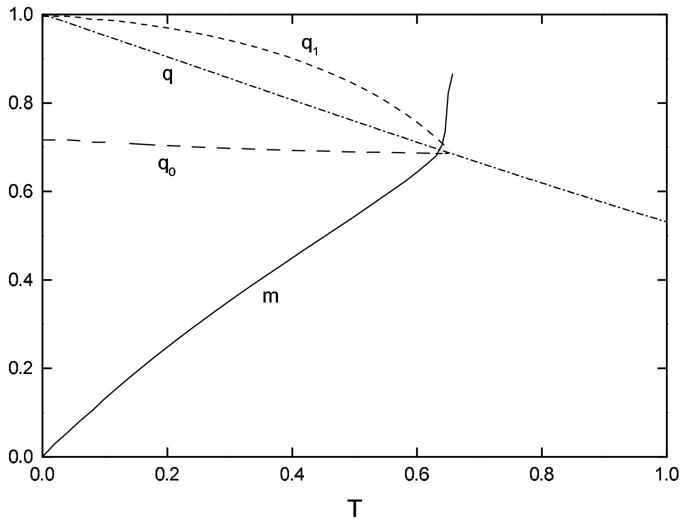

Figure 3. The same as figure 2 but for $h = 1$. The continuous transition occurs at $T \approx 0.66$, at which $m \approx 0.87$.

3. Transition lines

As indicated in the figures presented before, there are two qualitatively different types of transition between the RS and the 1RSB regimes which we will discuss separately in the following.

3.1. Continuous transition line

The location of the continuous transition between the RS and the 1RSB solution is determined by solving the 1RSB equations in the limit of small $q_1 - q_0$. More pointedly, subtracting equation (20) from (21) and keeping terms up to the order $(q_1 - q_0)^2$ yields

$$q_1 - q_0 = \frac{2q_0^2}{\beta^2(p-1)\lambda_0} \frac{B_0(q_0)}{B_2(q_0, m)} \tag{31}$$

where

$$B_0(q_0) = 1 - \beta^2(p-1)\frac{\lambda_0}{q_0}\int_{-\infty}^{\infty} \mathrm{D}z\ \mathrm{sech}^4[\beta\,\Xi_0(z)] \tag{32}$$

and

$$\begin{aligned} B_2(q_0, m) = {} & [p - 2 + 4\beta^2(p-1)\lambda_0(3-2m)]\int_{-\infty}^{\infty} \mathrm{D}z\ \mathrm{sech}^4[\beta\,\Xi_0(z)] \\ & -2\beta^2(p-1)\lambda_0(8-5m)\int_{-\infty}^{\infty} \mathrm{D}z\ \mathrm{sech}^6[\beta\,\Xi_0(z)]. \end{aligned} \tag{33}$$

Here

$$\Xi_0(z) = z\sqrt{\lambda_0} + h \tag{34}$$

with λ_0 given by equation (19). We note that both B_0 and B_2 are negative quantities in the 1RSB regime. Since at the continuous transition $q_1 \to q_0 \to q$, where q is the RS saddle-point (13), the transition line is given by the condition

$$B_0(q) = 0 \tag{35}$$

which, as expected, coincides with the RS stability line given by equation (14). To specify the value of m at the critical line, denoted by m_c, we expand equation (22) for small $q_1 - q_0$ (in this case we must keep terms up to the order $(q_1 - q_0)^3$) and then subtract it from equation (20). Using the condition (35) together with $q_0 \to q$ yields

$$1 - m_c = -\frac{B_2(q, 1)}{B_4(q)} \tag{36}$$

where

$$B_4(q) = 12(p-2) \int_{-\infty}^{\infty} \mathrm{D}z\ \mathrm{sech}^2[\beta \Xi_s(z)] \tanh^2[\beta \Xi_s(z)] \\ +2\beta^2(p-1)\lambda \int_{-\infty}^{\infty} \mathrm{D}z\ \mathrm{sech}^6[\beta \Xi_s(z)] \tag{37}$$

with Ξ_s and λ given by equations (10) and (12), respectively. Equation (36) holds provided that $m_c \leqslant 1$ and so the continuous transition line must end at a tricritical point, whose location is obtained by solving

$$B_2(q, 1) = 0 \tag{38}$$

and equation (35) simultaneously. As usual, the denominator in equation (31) vanishes at the tricritical point.

3.2. *Discontinuous transition line*

The location of the discontinuous transition line is determined by equating the free energies of the RS and 1RSB solutions, given by equations (9) and (17), respectively. This task is greatly facilitated in this case by noting that setting $m = 1$ in equation (20) yields $q_0 = q$. Moreover, since for $m = 1$ equation (17) becomes independent of q_1 (and λ_1) one has $f_{rsb}(m = 1) = f_{rs}$. Thus, for fixed h the temperature at which the discontinuous transition takes place is obtained by solving the 1RSB saddle-point equations with $m = 1$ for q_1, $q_0 = q$, and $T = T_c$.

3.3. *Analysis of the results*

The phase diagrams in the plane (T, h) are presented in figures 4–6 for $p = 2$, 3 and 10, respectively. The full curves are the RS stability condition, equation (14), whose upper branch coincides with the continuous transition line, equation (35). The discontinuous transition lines (chain curves) join the continuous ones at the tricritical points (full circles). We also present the lines at which the entropy of the RS solution vanishes (short-broken curves), which for $h = 0$ intersect the temperature axis at $T = 1/(2\sqrt{\ln 2}) \approx 0.60$, whatever the value of $p > 2$. We note that for $p \to \infty$ the condition for the vanishing of the RS entropy yields exactly the discontinuous transition line for the random energy model, equation (24). The agreement between these lines is already very good for $p = 10$ and h not too near h_t, as illustrated in figure 6.

Since our results are valid for non-integer, though physically meaningless, values of $p \geqslant 2$ as well, in figure 7 we present the value of the saddle-point parameter m at the continuous transition line, given by equation (36), for several values of p. As expected, for $p > 2$ we find $m_c = 1$ at the tricritical points. We mention that, despite the numerous studies of the SK model, we are not aware of any calculation of the 1RSB saddle-point parameter m over the Almeida–Thouless stability line. In figures 8 and 9 we present the values of T and h at the tricritical point, respectively, as functions of the real variable p. For $p \to 2$ we find $h_t \to 0$ and $T_t \to 1$, while for large p we find that h_t increases like $\sqrt{p \ln p}$, T_t like $\sqrt{p/\ln p}$ and

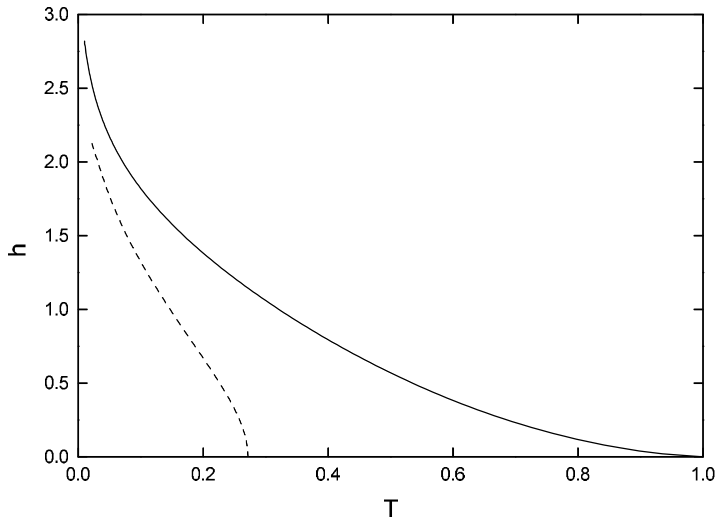

Figure 4. The phase diagram in the plane (T, h) for $p = 2$. The RS saddle-point is locally unstable inside the region delimited by the full curve, which coincides with the continuous transition line between the RS and the 1RSB regimes. The short-dashed curve delimits the region inside which the RS entropy is negative.

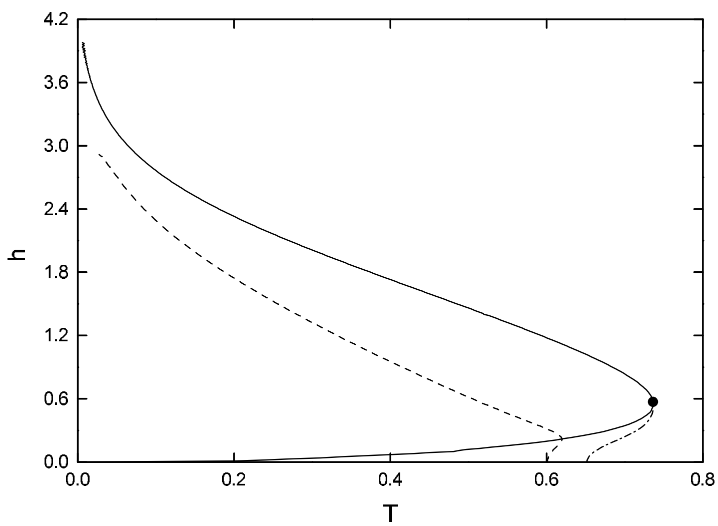

Figure 5. The same as figure 4 but for $p = 3$. The RS stability line coincides with the continuous transition line in the branch above the tricritical point (full circle), located at $T_t \approx 0.74$ and $h_t \approx 0.57$. The chain curve is the discontinuous transition line. The convention is the same as for figure 4.

$1 - q_t$ goes to zero like $1/(p\sqrt{\ln p})$. These results indicate that the phase diagrams in the plane (T, h) display the two types of transitions except in the extreme cases $p = 2$ and $p \to \infty$.

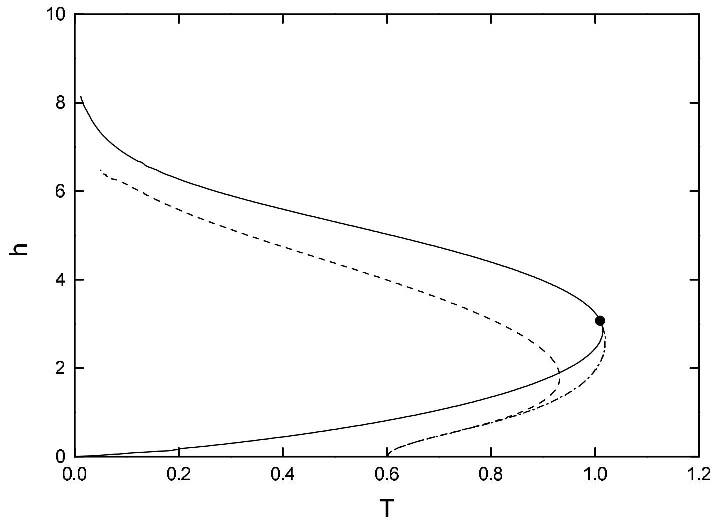

Figure 6. The same as figure 5 but for $p = 10$. The tricritical point (full circle) is located at $T_t \approx 1.01$ and $h_t \approx 3.07$.

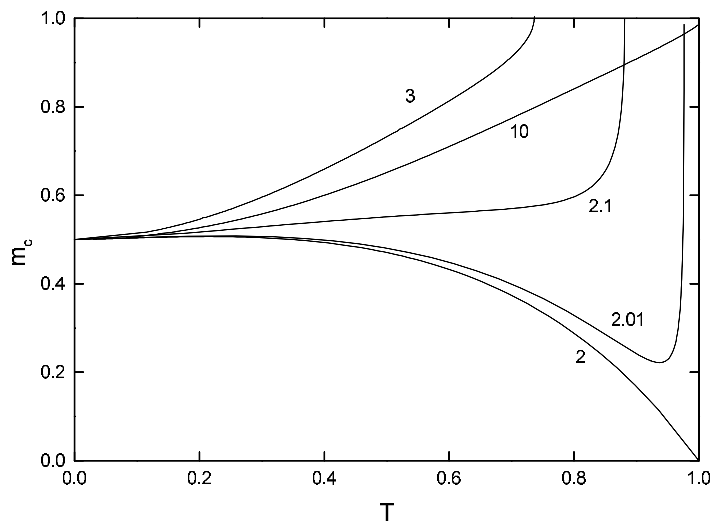

Figure 7. The saddle-point parameter m at the continuous transition line for (from bottom to top at $T = 0.5$) $p = 2$, 2.01, 2.1, 10 and 3.

4. Conclusion

Some comments regarding the validity of the 1RSB solution are in order. The stability analysis of that solution carried out for $h = 0$ indicates that it becomes unstable for low temperatures [11]. A seemingly simpler approach to check the physical soundness of the 1RSB solution is to numerically evaluate its entropy. This procedure, however, has proved very elusive: since the entropy becomes negative when it is of order $e^{-\beta p}$, the numerical precision required to determine the temperature T'' at which it vanishes is exceedingly large.

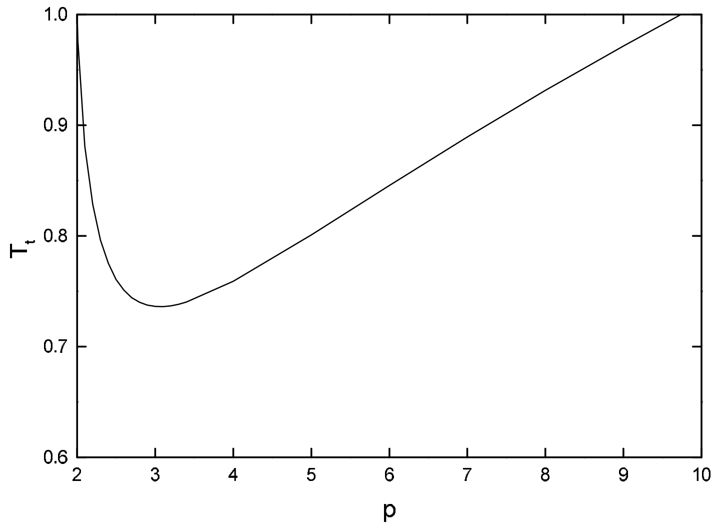

Figure 8. The temperature at the tricritical point T_t as a function of $p > 2$. Only integer values of p have a physical meaning.

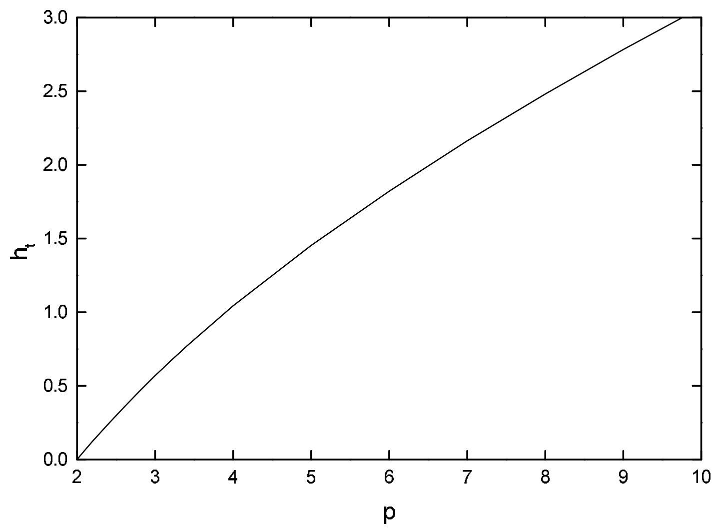

Figure 9. The magnetic field at the tricritical point h_t as a function of $p > 2$. Only integer values of p have a physical meaning.

For instance, for $h = 0$ we find $T'' = 0.10$, 0.087 (0.19) and 0.034 (0.18) for $p = 2$, 3 and 5, respectively. The numbers between parentheses are the numerical estimates of [12]. As our numerical results are in good agreement with that of [16] for $p = 2$, and are also consistent with the trend of decreasing T'' with increasing p, we think they are the correct ones. Already for $p > 5$, however, we have failed to obtain reliable estimates for T''. Since the precision problem becomes much worse for non-zero h, due to the numerical evaluation of the double integrals, we refrain from presenting the estimates for T'' in that case.

Although for finite p the 1RSB solution certainly does not correctly describe the low-temperature phase of the p-spin Ising spin glass, it probably yields the correct solution near the transition line delimiting the RS and the replica symmetry breaking regimes. In fact, according to Gardner [11], considering further steps of replica symmetry breaking within Parisi's scheme will result in a new *continuous* transition between the 1RSB regime and a more complex regime, described by a continuous order parameter function. In this sense, we think that our results regarding the transition lines between the RS and the 1RSB regimes are not mere artifacts of the replica method but indeed describe genuine features of the thermodynamics of the infinite range p-spin Ising spin glass in a magnetic field.

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